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Remark on General Lorentz-covariance

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Abstract

The present paper is a mathematical comment on a paper by Treder (Treder, 1970). The metric tensor g reduces the tangent bundle T of a space-time to the tangent Lorentz bundle L. The Levi-Civita connection in T induces in a natural way connections in L, and in the tensor product bundles of T, L, and the spinor bundle S. These connections are 'the general Lorentz-covariant connection' by Treder. It is possible to treat the local tetrad field components as local components of a global cross-section of $L \otimes T$, with vanishing general Lorentz-covariant derivative.

1. *The Tangent Lorentz Bundle and its Connection*

In general, a space-time manifold V_4 cannot be covered by one system of reference Σ. With V_4 assumed to be paracompact, it can be covered by a countable family $\mathscr{S}$ of systems of reference. Any two overlapping systems of reference, $\Sigma_{(\iota)}$, $\Sigma_{(\kappa)}$, are, on the overlap of their ranges, related by a field of Lorentz transformations $L_{(\iota\kappa)}$.

In fact, by means of an everywhere regular metric g, the structure group $GL(4, \mathbf{R})$ of the covariant tangent bundle T (i.e. the Einstein group) can be reduced to the Lorentz group. T looked at as a fibre bundle with Lorentz group as structure group (in short: as a Lorentz bundle), we denote by L. The family $\mathscr{S}$ is an admissible bundle atlas of L. The quantities $L_{(\iota\kappa)}$ form the corresponding system of coordinate transformations. Of course, $\mathscr{S}$ is also an admissible bundle atlas of T.

In terms of a so-called natural frame (that is a frame spanned by the differentials of local coordinates dx^l), the four legs of $\Sigma_{(\iota)}$ are

$$\Sigma_{(\iota)}{}^A = dx^l \underset{(\iota)}{h}{}_l^A, \qquad \text{inverse} \qquad dx^l = \Sigma_{(\iota)}{}^A \underset{(\iota)}{h}{}^l_A$$

with natural components $\underset{(\iota)}{h}{}_l^A$. Change of local coordinate implies transformation of natural vector components by an element of $GL(4, \mathbf{R})$: with $\mathbf{a} \in T$,

$$\mathbf{a} = dx^l a_l = dx^{l'}(x^l{}_{,l'}\, a_l), \qquad (x^l{}_{,l'}) \in GL(4, \mathbf{R})$$

i.e.

$$a_l \to a_{l'} = x^l{}_{,l'}\, a_l$$

Since

$$\mathbf{a} = dx^l a_l = \Sigma_{(\iota)}{}^A (\underset{(\iota)}{h}{}^l{}_A\, a_l) = \Sigma_{(\iota)}{}^A a_A$$

the Lorentz vector components are not affected by change of local coordinates. Change of Lorentz frames,

$$\Sigma_{(\iota)}{}^{A'} = \Sigma_{(\iota)}{}^A L_A^{A'}, \qquad (L_A^{A'}) \in \text{Lorentz group}$$

in particular the change of systems of reference where they overlap,

$$\Sigma_{(\iota)} = \Sigma_{(\kappa)} L_{(\kappa\iota)}$$

implies the transformation of Lorentz vector components by a Lorentz transformation:

$$\mathbf{a} = \Sigma_{(\iota)}{}^{A'} a_{A'} = \Sigma_{(\iota)}{}^A L_A^{A'} a_{A'} = \Sigma_{(\iota)}{}^A a_A,$$
$$\mathbf{a} = \Sigma_{(\iota)}{}^{A'} a_{A'} = \Sigma_{(\kappa)}{}^A (L_{(\kappa\iota)}{}_A^{A'} a_{A'}) = \Sigma_{(\kappa)}{}^A a_A$$

respectively.

The local components $\Gamma^A{}_{Bl}$ of a connection in a vector bundle (A, B are indices defined with respect to an admissible bundle chart; l is the label for the natural components) transform according to the law (Bräuer, 1970)

$$\Gamma^{A'}{}_{B'l'} = x^l{}_{,l'} L_A^{A'} L_{B'}^B \Gamma^A{}_{Bl} + L_A^{A'} L_{B',l'}^A \qquad \text{with} \qquad L_A^{A'} L_{B'}^A = \delta_{B'}^{A'}$$

where $(x^l{}_{,l'})$ corresponds to a transformation of local coordinates (from one admissible chart for V_4 to another, on their overlap) and $(L_A^{A'})$ is an element of the structure group translating local vector components from one admissible bundle chart to another.

For natural bundle charts, in the tangent bundle, with minuscules in place of capitals, i.e. with structure group $GL(4,\mathbf{R})$, this equation reads

$$\Gamma^{i'}{}_{k'l'} = x^l{}_{,l'} x^{i'}{}_{,i} x^k{}_{,k'} \Gamma^i{}_{kl} + x^{i'}{}_{,i} x^i{}_{,k'l'}$$

Without change of local coordinates, i.e. if $x^l = \delta^l_{l'} x^{l'}$, it reads

$$\Gamma^{A'}{}_{B'l} = L_A^{A'} L_{B'}^B \Gamma^A{}_{Bl} + L_A^{A'} L_{B',l}^A \tag{1.1}$$

The translation of natural components into Lorentz components is done by elements $(h_l^A) \in GL(4,\mathbf{R})$. In this case, (1.1) turns into the Lorentz affinity

$$\Gamma^A{}_{Bl} = h_k^A h_B^m \Gamma^k{}_{ml} + h_m^A h_{B,l}^m =$$
$$= h_m^A (h_B^k \Gamma^m{}_{kl} + h_{B,l}^m) = h_m^A h_{B;l}^m$$

and we notice that the Lorentz connection (defined by the Lorentz affinities) is not a second connection besides the connection defined by the Christoffel symbols. Lorentz and Christoffel affinities are the local components of one and the same connection only seen from different bundle atlasses.

2. *General Lorentz-covariant Connection*

The metric tensor g seen as a cross section of $T \otimes T$, has the (natural) components g_{ik}. Seen as a cross section of $L \otimes L$, its local components are η_{AB} with respect to any admissible bundle chart of $L \otimes L$:

$$g = (dx^i \otimes dx^k) g_{ik} = (\Sigma_{(\iota)}{}^A \underset{(\iota)}{h}{}^i_A \otimes \Sigma_{(\iota)}{}^B \underset{(\iota)}{h}{}^k_B) g_{ik} =$$
$$= (\Sigma_{(\iota)}{}^A \otimes \Sigma_{(\iota)}{}^B) \underset{(\iota)}{h}{}^i_A \underset{(\iota)}{h}{}^k_B g_{ik} = (\Sigma_{(\iota)}{}^A \otimes \Sigma_{(\iota)}{}^B) \eta_{AB}$$

As fibre bundles, T and L are, of course, not identical but $GL(4, \mathbf{R})$-equivalent. (The above terminology suggests calling this equivalence Einstein equivalence.) This is what Treder calls duality between the Lorentz covariant and the coordinate covariant representation of tensorial quantities.

The Levi-Civita connection in T with local components $\Gamma^i{}_{kl}$ induces a connection in the Lorentz bundle L with local components $\Gamma^A{}_{Bl}$. L is the natural residence of the connection defined by g because here the covariant constancy of g coincides with the constancy of its local components (η_{AB}). T together with the Levi-Civita connection is nothing but the $GL(4, \mathbf{R})$- or Einstein image of L and its resident.

It seems unusual to study something like ϕ_{Al}, since why should we describe one and the same object partly by this bundle chart and partly by that! But the bridge to the Einstein image T of L is built just with such hybrids as the half-transformed metric tensor

$$g_{kl} \underset{(\iota)}{h}{}^k_A = \underset{(\iota)}{g}_{Al} \equiv \underset{(\iota)}{h}_{Al} = \eta_{AB} \underset{(\iota)}{h}{}^B_l$$

called tetrad field. On the other hand, since these hybrids are elements of a tangent tensor bundle over V_4 (unusually looked at!), it could seem unreasonable to conduct parallel displacement with only one half of each. The complete parallel displacement is parallel displacement in $L \otimes T$ which is uniquely defined by the connections in L and T and, hence, by the connection in T alone. Explicit formulae can easily be obtained. The result is Treder's general covariant derivation (Treder, 1970).

It is not surprising to find that the general covariant derivatives of the tetrads vanish, because substantially the tetrad is the hybridized metric tensor:

$$(\Sigma_{(\iota)}{}^A \otimes dx^l) \underset{(\iota)}{h}_{Al} = (\Sigma_{(\iota)}{}^A \otimes dx^l \underset{(\iota)}{h}{}^B_l) \eta_{AB} = (\Sigma_{(\iota)}{}^A \otimes \Sigma_{(\iota)}{}^B) \eta_{AB}$$

and the $\underset{(\iota)}{h}_{Al}$ are the local components of a global cross-section of $L \otimes T$.

As a matter of course, we are not forced to handle the tetrad fields this way. For many purposes it will be more practical to take them as what they originally are: local vector fields on V_4.

3. *Connections on V_4 with Spinor Structure*

A spinor structure is a double covering of the principal Lorentz bundle $P(L)$ by a principal bundle $P(S)$ with the unimodular group $SL(2)$ as structure group. Its existence has topological implications on V_4. Since $SL(2)$ is used as covering group of the restricted Lorentz group, there is exactly one connection in $P(S)$ and its associated tensor bundles. The local components in the associated vector bundle S (= spinor bundle) are Iwanenko's spinor affinities. Again there is a unique ('general Lorentz-covariant') connection defined in the tensor product bundles formed with S, T, L (Bräuer, 1970; Treder, 1970).

According to Geroch's discovery (Geroch, 1968), however, the topological condition on a noncompact V_4 to admit a spinor structure is the existence of a Fernparallelismus on V_4. That is, such a space-time can be covered by one single system of reference Σ.

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